

# COMMUTING AND NON-COMMUTING INFINITESIMALS

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**ABSTRACT.** Infinitesimals are natural products of the human imagination. Their history goes back to the Greek antiquity. Their role in the calculus and analysis has seen dramatic ups and downs. They have stimulated strong opinions and even vitriol. Edwin Hewitt developed hyperreal fields in the 1940s. Abraham Robinson's infinitesimals date from the 1960s. A noncommutative version of infinitesimals, due to Alain Connes, has been in use since the 1990s. We review some of the hyperreal concepts, and compare them with some of the concepts underlying noncommutative geometry.

## 1. A BRIEF HISTORY OF INFINITESIMALS

A theory of infinitesimals was developed by Abraham Robinson in the 1960s (see [57], [59]). In France, Robinson's lead was followed by the analyst G. Choquet<sup>1</sup> and his group. Alain Connes started his work under Choquet's leadership. In 1970, Connes published two texts on hyperreals and ultrapowers [11, 12]. Some time after Robinson's death in 1974, Connes developed an alternative theory of infinitesimals. Connes' presentation of his theory is frequently accompanied by criticism of Robinson's.

Our goal is to compare the concept of an infinitesimal in Robinson's hyperreals, with an analogous concept in Connes' noncommutative geometry. We shall also review several comments by Connes on the subject of Robinson's hyperreals, such as the following comment made in 2000:

A nonstandard number is some sort of chimera which is impossible to grasp and certainly not a concrete object (Connes [17], [18]).

D. Tall describes a concept in cognitive theory he calls a *generic limit concept* in the following terms:

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<sup>1</sup>See e.g., Choquet's work on ultrafilters [10]. Choquet's constructions were employed and extended by Mokobodzki [55].

if a quantity repeatedly gets smaller and smaller and smaller without ever being zero, then the limiting object is naturally conceptualised as an extremely small quantity that is not zero (Cornu [22]). Infinitesimal quantities are natural products of the human imagination derived through combining potentially infinite repetition and the recognition of its repeating properties (Tall [69], [70]).

Leibniz based the fundamental concepts of the calculus on infinitesimals, an approach that was followed by l'Hôpital, Johann Bernoulli, Varignon, and others. Thus, the “differential quotient” (which evolved into the modern derivative) was thought of as a ratio of infinitesimals, while the integral, as an infinite sum of areas of infinitely narrow rectangles.

In the context of European mathematics, infinitesimals were already uppermost in the mind of Nicholas of Cusa in the 15th century. Nicholas of Cusa thought of the circle as a polygon with infinitely many sides, inspiring Kepler to formulate his “bridge of continuity” (see Baron [5, p. 110]). The law of continuity was a heuristic principle developed by Leibniz. He formulated it as follows in a 1702 letter to Varignon:

The rules of the finite succeed in the [realm of the] infinite [...] and vice versa the rules of the infinite succeed in the [realm of the] finite (Leibniz 1702, [48]; cf. Robinson [59, p. 262, 266]; Knobloch [46, p. 67]).

Robinson put it as follows in 1966:

Leibniz did say [...] that what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa, and this is remarkably close to our transfer of statements from  $\mathbb{R}$  to  ${}^*\mathbb{R}$  and in the opposite direction. But to what sort of laws was this principle supposed to apply? [59, p. 266] (see also Laugwitz [47]).

A century after Leibniz, L. Carnot and A.-L. Cauchy still exploited the concept of an infinitesimal, generated by a suitable variable quantity, namely a null sequence (sequence tending to zero). An alternative foundation for the calculus in terms of the epsilon, delta approach was developed by G. Cantor, R. Dedekind, and K. Weierstrass starting in the 1870s. As an example of the epsilon-delta method, consider Cauchy's definition of continuity of a function  $y = f(x)$ :

*an infinitesimal change  $\alpha$  of the independent variable  $x$   
always produces an infinitesimal change  $f(x + \alpha) - f(x)$   
of the dependent variable  $y$  (cf. Cauchy [9, p. 34]).<sup>2</sup>*

Weierstrass reconstructs Cauchy’s infinitesimal definition as follows: for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that for every  $\alpha$ , if  $|\alpha| < \delta$  then  $|f(x + \alpha) - f(x)| < \epsilon$ .

Before infinitesimals would finally be justified in a mathematically rigorous fashion by Robinson [59], they were yet to be derided as “paper numbers”, “cholera bacillus” of mathematics, and an “abomination” by Georg Cantor (see Meschkowski [53, p. 505]), who had convinced himself of the impossibility of justifying infinitesimals in such a fashion (see J. Dauben [25], P. Ehrlich [29], and Błaszczyk et al. [6] for details).

## 2. ROBINSON’S FRAMEWORK

In 1961, Robinson [57] constructed an infinitesimal-enriched continuum, suitable for use in calculus, analysis, and elsewhere, based on earlier work by E. Hewitt [34], J. Łoś [50], and others. In 1962, W. Luxemburg [51] popularized a presentation of Robinson’s theory in terms of the ultrapower construction,<sup>3</sup> in the mainstream foundational framework of the Zermelo–Fraenkel set theory with the axiom of choice (ZFC). Namely, the hyperreal field is the quotient of the collection of arbitrary sequences, where a sequence

$$\langle u_1, u_2, u_3, \dots \rangle \tag{2.1}$$

converging to zero generates an infinitesimal. Arithmetic operations are defined at the level of representing sequences; e.g., addition and multiplication are defined term-by-term.

To motivate the construction of the hyperreals, note that the construction can be viewed as a relaxing, or refining, of Cantor’s construction of the reals. This can be motivated by a discussion of rates of convergence as follows. In Cantor’s construction, a real number  $u$  is represented by a Cauchy sequence  $\langle u_n : n \in \mathbb{N} \rangle$  of rationals. But the passage from  $\langle u_n \rangle$  to  $u$  in Cantor’s construction sacrifices too much information. We would like to retain a bit of the information about the sequence, such as its “speed of convergence”. This is what one means by “relaxing” or “refining” Cantor’s construction (cf. Giordano

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<sup>2</sup>For details on Cauchy’s infinitesimals, see Cutland et al. [23], Błaszczyk et al. [6], Borovik et al. [8].

<sup>3</sup>Note that both the term “hyper-real”, and an ultrapower construction of a hyperreal field, are due to E. Hewitt in 1948, see [34, p. 74]. Luxemburg [51] also clarified its relation to the competing construction of Schmieden and Laugwitz [63], similarly based on sequences, which used a different kind of filter.

et al. [33]). When such an additional piece of information is retained, two different sequences, say  $\langle u_n \rangle$  and  $\langle u'_n \rangle$ , may both converge to  $u$ , but at different speeds. The corresponding “numbers” will differ from  $u$  by distinct infinitesimals. If  $\langle u_n \rangle$  converges to  $u$  faster than  $\langle u'_n \rangle$ , then the corresponding infinitesimal will be smaller. The retaining of such additional information allows one to distinguish between the equivalence class of  $\langle u_n \rangle$  and that of  $\langle u'_n \rangle$  and therefore obtain different hyperreals infinitely close to  $u$ .

At the formal level, we proceed as follows. We start with the ring  $\mathbb{Q}^{\mathbb{N}}$  of sequences of rational numbers. Let

$$\mathcal{C}_{\mathbb{Q}} \subset \mathbb{Q}^{\mathbb{N}} \quad (2.2)$$

denote the subring consisting of Cauchy sequences. The reals are by definition the quotient field

$$\mathbb{R} := \mathcal{C}_{\mathbb{Q}} / \mathcal{F}_{null}, \quad (2.3)$$

where  $\mathcal{F}_{null}$  contains all null sequences (i.e., sequences tending to zero). An infinitesimal-enriched extension of  $\mathbb{Q}$  may be obtained by modifying (2.3) as follows. We consider a subring  $\mathcal{F}_{ez} \subset \mathcal{F}_{null}$  of sequences that are “eventually zero”, i.e., vanish at all but finitely many places. Then the quotient  $\mathcal{C}_{\mathbb{Q}} / \mathcal{F}_{ez}$  naturally surjects onto  $\mathbb{R} = \mathcal{C}_{\mathbb{Q}} / \mathcal{F}_{null}$ . The elements in the kernel of the surjection

$$\mathcal{C}_{\mathbb{Q}} / \mathcal{F}_{ez} \rightarrow \mathbb{R}$$

are prototypes of infinitesimals. Note that the quotient  $\mathcal{C}_{\mathbb{Q}} / \mathcal{F}_{ez}$  is not a field, as  $\mathcal{F}_{ez}$  is not a maximal ideal. The natural next step is to replace  $\mathcal{F}_{ez}$  by a *maximal* ideal. It is more convenient to describe the modified construction using the ring  $\mathbb{R}^{\mathbb{N}}$  rather than  $\mathcal{C}_{\mathbb{Q}}$  of (2.2).

We therefore redefine  $\mathcal{F}_{ez}$  to be the ring of real sequences in  $\mathbb{R}^{\mathbb{N}}$  that eventually vanish, and choose a *maximal* proper ideal  $\mathcal{M}$  so that we have

$$\mathcal{F}_{ez} \subset \mathcal{M} \subset \mathbb{R}^{\mathbb{N}}. \quad (2.4)$$

Then the quotient  $\mathbb{R} := \mathbb{R}^{\mathbb{N}} / \mathcal{M}$  is a hyperreal field. The foundational material needed to ensure the existence of a maximal ideal  $\mathcal{M}$  satisfying (2.4) is weaker than the axiom of choice. This concludes the construction of a hyperreal field  $\mathbb{R}$  (“thick- $\mathbb{R}$ ”) in the traditional foundational framework, ZFC.

Let  $I \subset \mathbb{R}$  be the subring consisting of infinitesimal elements (i.e., elements  $e$  such that  $|e| < \frac{1}{n}$  for all  $n \in \mathbb{N}$ ). Denote by  $I^{-1}$  the set of inverses of nonzero elements of  $I$ . The complement  $\mathbb{R} \setminus I^{-1}$  consists of all the finite (sometimes called *limited*) hyperreals. Constant sequences provide an inclusion  $\mathbb{R} \subset \mathbb{R}$ . Every element of the complement,  $x \in$

$\mathbb{R} \setminus I^{-1}$  is infinitely close to some real number  $x_0 \in \mathbb{R}$ . The *standard part function*, denoted “st”, associates to every finite hyperreal, the unique real infinitely close to it:

$$\text{st} : \mathbb{R} \setminus I^{-1} \rightarrow \mathbb{R}, \text{ with } x \mapsto x_0.$$

If  $x$  happens to be represented by a Cauchy sequence  $\langle x_n : n \in \mathbb{N} \rangle$  then  $x_0 = \text{st}(x) = \lim_{n \rightarrow \infty} x_n$ . More advanced properties of the hyperreals such as saturation were proved later (see Keisler [45] for a historical outline). A helpful “semicolon” notation for presenting an extended decimal expansion of a hyperreal was described by A. H. Lightstone [49]. See also P. Roquette [62] for infinitesimal reminiscences. A discussion of infinitesimal optics is in K. Stroyan [67], J. Keisler [44], D. Tall [68], and L. Magnani and R. Dossena [52, 27]. Applications of infinitesimal-enriched continua range from aid in teaching calculus [31, 36, 37] to the Boltzmann equation (see L. Arkeryd [3, 4]), mathematical physics (see Albeverio *et al.* [1]); etc. Edward Nelson [56] in 1977 proposed an alternative to ZFC which is a richer (more stratified) axiomatisation for set theory, called Internal Set Theory (IST), more congenial to infinitesimals than ZFC. A rethinking of the history, mathematics, and philosophy of infinitesimals has been undertaken in [6], [7], [8], [38], [39], [40], [41], [42], [43]. Recently, P. Ehrlich [30, Theorem 20] showed that the ordered field underlying a maximal (i.e., *On*-saturated) hyperreal field is isomorphic to J. H. Conway’s ordered field No, an ordered field Ehrlich describes as the *absolute arithmetic continuum*.

### 3. FROM PHYSICS TO NONCOMMUTING INFINITESIMALS

One of the revolutionary observations of 20th century physics is that observables cannot take scalar values. Experiments have shown that position  $X$  and momentum  $P$  satisfy the relation (frequently described as the uncertainty principle)

$$[X, P] = i\hbar,$$

which expresses such an absence of commutation.

Since addition and multiplication of sequences is term-by-term, the multiplication discussed in Section 1 is still commutative. Now think of the sequence (2.1) as a diagonal matrix

$$\text{diag}(u_1, u_2, u_3, \dots)$$

of infinite size. We now enlarge the collection of matrices, by propagating by an action of  $\mathrm{SO}(\infty)$ . We obtain a vast collection of noncommuting matrices. The original sequence can be retrieved by symmetrisation, followed by applying Spec (the spectrum, i.e. calculating the eigenvalues).

At this point, a mathematician worth his ilk suppresses the dependence on the preferred basis, spruces up *matrices* as *operators*, and recalls a basic fact from functional analysis: an operator whose  $n$ -th eigenvalue tends to zero, is a compact operator (image of the unit ball is compact).

In practice, Connes chooses a much smaller collection of compact operators as his class of infinitesimals. It is defined by sequences with a specific rate of convergence to 0.

Connes published two papers [11, 12] on non-standard analysis in 1970. The paper [11] is specifically devoted to the ultrapower approach. The connection between Connes' infinitesimals and Robinson's via the sequence defined by the spectrum is an intriguing one, see Albeverio et al. [2].<sup>4</sup>

#### 4. NONCOMMUTATIVE GEOMETRY AND NON-STANDARD ANALYSIS

Noncommutative geometry is one of the fastest growing theories today, influential both in mathematics and in physics. One of the technical aspects of noncommutative geometry, which plays a role, in particular, in establishing a mathematical framework for high-energy physics, is the Dixmier trace.

Both Connes' and Robinson's theories involve a concept of an infinitesimal, leading to a natural question as to the comparison of the two.

Note that a compact operator is by definition one whose  $n$ -th eigenvalue tends to zero (at least modulo symmetrisation). This motivates viewing a compact operator as an infinitesimal (see Section 3).

Connes exploited the Dixmier trace [26] to give a uniform explanation of the pseudodifferential residue<sup>5</sup> of Guillemin and Wodzicki. The result was immediately hailed as a major accomplishment. The exotic traces constructed by J. Dixmier are not normal, and have the property of being nonnegative on a compact positive operator. In his

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<sup>4</sup>Yamashita [72, 71] applied Robinson's hyperreals to quantum field theory, but did not emphasize the relation to Connes' noncommutative geometry.

<sup>5</sup>No understanding of the pseudodifferential residue is required for understanding the present text.

article *Brisure de symétrie*, Connes argues that his solution is *substantial and calculable* [14, p. 211], and expresses a disappointment with an allegedly non-exhibitable nature of Robinson's infinitesimals. Meanwhile, Dixmier's construction of the trace relies on the choice of an ultrafilter on the integers.<sup>6</sup>

It should be stated at the outset that the focus of interest of Connes' noncommutative geometry lies elsewhere. Still, one of the building blocks of Connes' theory is a framework incorporating non-commuting infinitesimals. In his articles published in refereed journals, Connes repeatedly stresses the role of Dixmier's traces in implementing a general framework for working with his infinitesimals. It is significant to ponder the fact itself of the existence of Connes' implementation, based, as it is, on Dixmier traces, whose foundational status is dependent on a choice of an ultrafilter, closely related to the axiom of choice.

While it has been pointed out that in concrete instances, the Dixmier trace is replaced by more concrete objects whose foundational status is more constructive (such a claim ought to be taken with a grain of salt, as developments in functional analysis frequently involve an unspoken dependence on such results as the Hahn-Banach theorem,<sup>7</sup> similarly non-constructive; be that as it may), it is significant that Connes indisputably succeeds in implementing a general framework based on Dixmier traces, inspite of their non-constructive foundational status.

In this context, it is instructive to examine Connes' published comments on Robinson's theory, which typically go hand in hand with Connes' acknowledgment of indebtedness to Dixmier and his traces.<sup>8</sup> The importance of Dixmier traces in noncommutative geometry was noted by S. Albeverio et al. [2].

## 5. CONNES' THREE OBJECTIONS TO ROBINSON'S THEORY

Connes has expressed himself on a number of occasions on the subject of Robinson's theory. Thus, in a 2000 interview, he said:

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<sup>6</sup>Connes has shown that a theorem of Mokobodzki (see [54]) provides a limit process (for the Dixmier trace) which is universally measurable, while relying on the continuum hypothesis (no understanding of either universal measurability or the continuum hypothesis is required in the sequel). See Remark 5.1 for more details.

<sup>7</sup>See Remark 5.1 illustrating Connes' reliance on the Hahn-Banach theorem.

<sup>8</sup>Note that Robinson studied generalized limits in 1964, in the context of (infinite) real Toeplitz matrices (see [58]); however a key property of *scale invariance* is apparently not present.

I became aware of an absolutely major flaw in this theory, an irremediable defect (Connes [21, p. 16]).

In 2007, Connes worded himself in a more nuanced way:

it seemed utterly doomed to failure to try to use non-standard analysis to do physics (Connes [19, p. 26]),

apparently implying that the alleged shortcomings of Robinson's theory are limited specifically to potential applications in *physics*.<sup>9</sup>

In 1995, Connes gave a detailed account of the role of the Dixmier trace in his theory. Connes states that the goal [16, section II] is to develop a “calculus of infinitesimals” based on operators in Hilbert space (see Section 3 above), and proceeds to

explain why the formalism of nonstandard analysis is inadequate (Connes [16, p. 6207]).

Connes points out the following three aspects of Robinson's hyperreals (the explicit list is ours, not Connes’):

- (1) a nonstandard hyperreal “cannot be exhibited” (the reason given being its relation to non-measurable sets);
- (2) “the practical use of such a notion is limited to computations in which the final result is independent of the exact value of the above infinitesimal. This is the way nonstandard analysis and ultraproducts are used”;
- (3) the hyperreals are commutative.

We will argue that two of the three arguments given by Connes with regard to the inadequacy of Robinson's theory may be weaker than he claims since they appear to apply similarly to his own calculus. If so, Connes' claim that his “theory of infinitesimal variables is completely different” may be exaggerated. For a sequel paper, see (Katz, Kanovei, Mormann 2013 [35]).

Connes proceeds to establish a dictionary on page 6208, relating classical and quantum notions. The last quantum item in his dictionary is the Dixmier trace, corresponding to “integrals of infinitesimals of order 1”. On pages 6210-6211, Connes presents a pair of technical difficulties with the theory, and states:

Both of these problems are resolved by the Dixmier trace (Connes [16, p. 6211]).

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<sup>9</sup>Recently, S. Kutateladze reacted as follows to Connes' comment: “*Physics* meant *mechanics* for about two centuries. Newton and Leibniz and their followers used heuristic infinitesimals, which were ultimately implemented mathematically by Robinson. Celestial mechanics and hydrodynamics still require infinitesimals in much the same way as in Leibniz.”



On page 6212, the Dixmier trace  $\text{Tr}_\omega$  is defined as any limit point of suitable functionals, where “the choice of the limit point is encoded by the index  $\omega$ ”. Connes goes on to state that

for measurable operators  $T$ , the value of  $\text{Tr}_\omega(T)$  is *independent* of  $\omega$  and this common value is the appropriate integral of  $T$  in the new calculus (Connes [16, p. 6213])  
[emphasis added—the authors]

Such *independence* would seem to relativize the impact of the objection (2) raised above, affirming precisely a similar independence, for  $\text{Tr}_\omega$ .

An alternative construction of the trace by Connes, while not explicit as it relies on the axiom of choice, is satisfactory since it is given by a limit process which is universally measurable (see [54]); however the construction depends on the continuum hypothesis (see Remark 5.1). The foundational status of the trace tends to relativize the objection (1).

Note that in Connes’ approach, an infinitesimal is given by a compact operator, and of course many of them can be naturally exhibited (this is at variance with the general framework, since in order to define, via the Dixmier trace, an interesting trace of the space of infinitesimals of degree 1, one exploits the axiom of choice). Similarly, many Robinson infinitesimals can also be naturally exhibited (see Section 6). Thus, two-thirds of Connes’ critique of Robinson’s infinitesimal approach can be said to be incoherent, in the specific sense of not being coherent with what Connes writes (approvingly) about his own infinitesimal approach. The remaining objection (3), namely the non-commutativity of the hyperreals, is by far the most convincing of the three objections. As Connes wrote, “The uniqueness of the separable infinite dimensional Hilbert space cures that problem, and variables with continuous range coexist happily with variables with countable range, such as the infinitesimal ones. The only new fact is that they do not commute” (Connes [20, Section 2]).

Note that the fact of working with non-commutative algebras allows Connes to deal with many interesting applications: foliations, set of irreducible representations of a noncommutative discrete group, etc.

In Connes’ approach, noncommutativity ensures the coexistence of variables having a Lebesgue spectrum with infinitesimal variables. To elaborate, consider the map  $x \rightarrow f(x)$  from  $[0, 1]$  into itself, where  $f$  is monotone increasing. By multiplication it defines a self adjoint operator from  $L^2([0, 1])$  into itself with Lebesgue spectrum. Now consider a map  $x \rightarrow A(x)$  from  $[0, 1]$  into itself having only a countable set of

values, for instance, the set  $\{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\}$ . Then, for infinitely many values of  $n$ ,  $A^{-1}(\{1/n\})$  is uncountable with positive measure. For such  $n$ ,  $1/n$  is an eigenvalue of the multiplication operator by  $A$  whose eigenspace is infinite dimensional. Thus,  $A$  cannot define an infinitesimal. In the noncommutative picture, a variable is given by a self adjoint operator of  $BL^2([0, 1])$ , and an infinitesimal is given by a compact self adjoint operator  $K$  of  $BL^2([0, 1])$ . The set of eigenvalues of  $K$  forms a sequence, it tends to zero and the eigenspaces associated to nonzero eigenvalues are finite dimensional.

The remarkable coincidence of dates: both Robinson's book [59] and Dixmier's article [26] were published in 1966, suggests that a certain cross-fertilisation of ideas may have taken place. Can this particular subparagraph of the early version of Noncommutative geometry, as it appeared in the nineties, be thought of having an important precursor in Robinson's monumental work [59], in addition to Dixmier's 2-page article [26]?

**Remark 5.1.** On pages 303-308 of his book [15], Connes presents a detailed construction of the Dixmier trace. On page 305, he chooses a positive linear form  $L$  on the vector space of bounded continuous functions on  $\mathbb{R}_+ \setminus \{0\}$ , such that  $L(1) = 1$ , and which is zero on the subspace of functions vanishing at infinity. The construction of the trace is eventually proved to be independent of such a choice. The choice of  $L$  relies on the Hahn-Banach theorem, of similar foundational status.

On the other hand, using a theorem of Mokobodzki for the Cantor set  $\{0, 1\}^{\mathbb{N}}$ , Connes constructed a Dixmier trace by a limit process which is universally measurable, based on foundational material including the continuum hypothesis (CH) (though it is impossible to exhibit a representative of this Dixmier trace). However, CH is generally considered to be a strong foundational hypothesis. Thus, all ultrapower-type models of the hyperreals starting from  $\mathbb{R}^{\mathbb{N}}$  become isomorphic if one assumes CH (see [28]).

## 6. EXHIBITABLE ROBINSON INFINITESIMAL

On the connection between infinitesimals and non-measurable sets, Connes writes as follows:

Every non-standard real determines canonically a Lebesgue non-measurable subset of the interval  $[0, 1]$ , so that it is impossible to **exhibit** a single one (Connes [14, p. 211]) [emphasis added—authors].

The expression “single one” apparently refers to “non-standard real”, and not “non-measurable set”, and has been widely interpreted as such in the literature.

Thus, as recently as 2009, M. de Glas claims as a matter of fact that each hyperreal is canonically associated with a non-measurable subset of the real line [24, p. 194].

Meanwhile, Keisler’s *Elementary calculus* [44] on page 913, line 3, **exhibits** an explicit representative of an equivalence class defining an infinitesimal hyperreal (in the context of the ultrapower construction following Luxemburg). Are they exhibitable or non-exhibitable? The explanation of the apparent paradox is as follows.

Given an infinite hypernatural  $H$ , we can consider all subsets  $A \subset \mathbb{N}$  whose natural hyperreal extension  ${}^*A \subset {}^*\mathbb{N}$  contains  $H$ . The resulting collection defines an ultrafilter on  $\mathbb{N}$  which, viewed as a subset of  $[0,1]$  via dyadic representation, produces “canonically” a non-measurable set. Similarly, given an arbitrary infinite hyperreal  $H$ , we can consider its integer part<sup>10</sup>

$${}^*[H], \quad (6.1)$$

and proceed “canonically” as before. Given an infinitesimal  $\epsilon$ , we can consider the hyperreal

$$H = \frac{1}{\epsilon} \quad (6.2)$$

and proceed “canonically” as in (6.1). Finally, given a non-standard finite hyperreal  $x$ , we can consider the infinitesimal difference

$$\epsilon = x - \text{st}(x), \quad (6.3)$$

where “st” is the standard part function (see Section 2), and proceed “canonically” as in (6.2).

The catch (hitch?) is implicit in the meaning of the word “canonical”. The construction is only canonical once one has in place the powerful new principles of reasoning such as the transfer principle (a mathematical implementation of Leibniz’s heuristic law of continuity), the standard part function, etc., so that it is possible to talk about natural extensions of standard objects.

Thus, while it is possible to exhibit a representative of a Robinson infinitesimal, the choices involved in constructing the hyperreals make the corresponding “canonical” non-measurable set, in fact nonexhibitable.

In the presence of an ultrafilter construction of the hyperreal line (see Section 2 above as well as (Keisler [44, p. 911])), one can prove the following result.

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<sup>10</sup>Here  ${}^*[H]$  is the image of  $H$  under  ${}^*[\ ]$ , the natural extension of the integer part function  $[\ ]$  on  $\mathbb{R}$ .

**Theorem 6.1.** *A choice of a Connes infinitesimal canonically produces a non-measurable set. Such a set cannot be exhibited [66].*

*Proof.* A Connes infinitesimal is a compact operator  $T$ . The Dixmier trace is applied to the spectrum of  $|T|$ . The spectrum can be canonically ordered in decreasing order of  $|\lambda_i|$ , producing a sequence  $(\lambda_i)$  which tends to zero. As such, it represents a Robinson infinitesimal [44, p. 911], and we proceed as in (6.2).  $\square$

#### ACKNOWLEDGMENT

We are grateful to V. Kanovei, S. Shelah, and D. Sherry for helpful comments.

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